where $S(x, 0) = 0$ and l in α_S needs to be interpreted as the latent heat of vaporization per unit mass.

As an example we consider a pulse with shape given by

$$Q(x, t) = Q_0 \delta(x)$$

where Q_0 is a constant. The integration of equation (14) produces

$$\frac{S}{S_{max}} = \left\{ \exp\left[-\frac{Q_0^2}{4\pi} X^2 \right] - \frac{Q_0}{2} |X| \operatorname{erfc}\left[\frac{Q_0 |X|}{(4\pi)^{\frac{1}{2}}} \right] \right\}, \quad (15)$$

where $X = x/S_{max}$ and $S_{max} = S(0, t) = Q_0(\alpha_S t/\pi)^{\frac{1}{2}}$.

For large values of $Q_0|X|/2\pi^{\frac{1}{2}}$, the solution (15) becomes [4]

$$\frac{S}{S_{max}} \simeq \frac{2\pi}{Q_0^2 X^2} \exp\left(-\frac{Q_0 X^2}{4\pi} \right) \simeq \frac{2\pi}{Q_0^2 X^2}.$$

Therefore if the surface aperture is measured by the distance at which $S/S_{max} = \varepsilon^2$ = a small constant, then the radius of the aperture is given by

$$x_\varepsilon \simeq \frac{\sqrt{(2)}}{\varepsilon} (\alpha_S t)^{\frac{1}{2}}.$$

The hole spreads along the surface proportional to $t^{\frac{1}{2}}$. The same conclusion is reached by Boley and Yagoda [1] from their early time solutions. It is also seen that the spreading is a function of α_S only.

$$\alpha_s = \frac{K(T_v - T_0)}{\rho[C_p(T_v - T_0) + l_v(1 - 1/n)]}$$

where T_v is the temperature of boiling point and l_v the latent heat of vaporization per unit mass.

However, contrary to Boley and Vagoda [1], the maximum penetration into the solid is also proportional to $t^{\frac{1}{2}}$. For example, in aluminum, a crater depth of one millimeter may be produced by a millisecond-duration pulse with the heat flux of 15 kW/cm. This value falls in the range of representative values given in Ready [5].

The shape of holes must be computed with a much more realistic heat input than the delta function, but this may require a numerical integration of equation (14). If equation (15) is used, then the slope of the hole at $X = 0 = -Q_0/2$ may be used as a measure for the steepness of the hole. For aluminum and the heat flux of 15 kW/cm, the slope is about -3. This is again comparable with a configuration of hole shown in Ready [5].

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INFLUENCE OF THERMAL PROPERTIES ON FILM COOLING EFFECTIVENESS

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INTRODUCTION

The recent publication of *Analysis of Heat and Mass Transfer* [1] and the re-introduction of a semi-analytical analysis of the film cooling effectiveness of [2] is the motivation for these authors to present certain aspects of the problem not yet available in the literature. Although the effectiveness presented in [1, 2] yields satisfactory agreement with experimental data, it is questionable as to whether the influence of variable fluid properties is expressed correctly [1]. In the present investigation, an attempt is made to study the influencing effect of the fluid properties in a purely analytical manner. The only empirical relation considered is the well-known Prandtl equation relating shear stress to momentum thickness. Also, it has been assumed that a power law relation $u/u_e = (y/\delta)^{1/n}$ for the velocity distribution in the boundary layer holds far downstream from the slot.

The nomenclature utilized are those of [1], and are mentioned here in the text. The film cooling effectiveness is expressed by

$$\eta = \frac{1/\lambda}{1+\frac{c_{p_e}}{c_{p_c}}\frac{m_e}{m_s}} \tag{1}$$

wherein subscripts e and c denote mainstream and coolant conditions respectively. The mass flux passing through the slot is m_s and the mass entrainment into the boundary layer from the mainstream is m_e. The value of $1/\lambda$ may be calculated by using the asymptotic behavior of equation (1) as [1]

$$1/\lambda \cong 1{\cdot}9\,Pr^{2/3}a \tag{2}$$

in which

$$a = \left[1-\left(\frac{L}{x-L}\right)^{0{\cdot}9}\right]^{-0{\cdot}8}.$$

It should be noted that the length L ahead of the slot is included in this analysis. However, in [1], it is assumed that $L = 0$ and consequently $a = 1$. Finally, the following relation for film cooling effectiveness is presented in reference [1] as

$$\eta = \frac{1{\cdot}9\,Pr^{2/3}}{1+0{\cdot}329\frac{c_{p_e}}{c_{p_c}}\xi\beta} \tag{3}$$

where

$$\xi = \left(\frac{\mu_e}{\mu_c}\right)^{0{\cdot}2}\left(\frac{\rho_e u_e x_1}{m_s}\right)^{0{\cdot}8} Re_s^{-0{\cdot}2} \tag{3a}$$

$$x_1 = x - s$$

and

$$\beta = 1 + 1{\cdot}5\times10^{-4}\,Re_s\left(\frac{\mu_c}{\mu_e}\right)\sin\alpha. \tag{3b}$$

The value of β was determined using available experimental data downstream from the slot [2]. It should be emphasized that $\beta > 1$ when x is finite. As x becomes large, the boundary layer thickness increases, and the value of β should approach unity. Therefore, β as an independent function of the coordinate x is a valid assumption only for a small range of x somewhere downstream from the slot.

ANALYSIS

By use of the momentum-integral theorem, the entrained mass is

$$m_e = \bar{\rho}_e u_e(\delta - \delta^*) - m_s, \tag{4}$$

where δ and δ^* are the boundary layer thickness and displacement thickness, and $\bar{\rho}_e$ is the average density within the boundary layer. Better accuracy may be obtained if $\bar{\rho}_e$ is calculated at temperature $\bar{T}$ using the equation [1]

$$\bar{T} = T_e + \frac{T_{Aw}-T_e}{1{\cdot}9\,Pr^{2/3}a}, \tag{5}$$

in which T_{Aw} is the adiabatic wall temperature and T_e is the mainstream temperature.

Integration of the momentum equation for $x > s$ (far downstream from the slot) yields

$$\theta(x) = \left[\frac{5}{4}\frac{0{\cdot}0128(x-s)}{(u_e/\nu)^{1/4}} + \left(\theta_L + \frac{m_s}{\rho_e u_e}\sin\alpha\right)^{5/4}\right]^{4/5}, \tag{7}$$

in which θ_L is the momentum thickness at the leading edge of the slot.

When x is large and $n = 7$, the effectiveness η becomes

$$\eta = \frac{1{\cdot}9\,Pr^{2/3}a}{1+\frac{c_{p_e}}{c_{p_c}}0{\cdot}329\,\xi\beta_1} \tag{8}$$

where

$$\beta_1 = \left[1+\left[\left(\frac{L}{x-s}\right)^{4/5} + \frac{27{\cdot}33}{Re_{x-s}^{4/5}}Re_s\sin\alpha\left(\frac{\mu_c}{\mu_e}\right)\right]^{5/4}\right]^{4/5}. \tag{8a}$$

The parametrical similarity between this equation and the effectiveness derived in [1] is apparent. The similarity becomes clearer for $L = 0$, in which case the variation of β_1 for no upstream boundary layer development ($L = 0$) is plotted in Fig. 1. For comparison, the variation of β, from equation (3b), has also been plotted in Fig. 1.

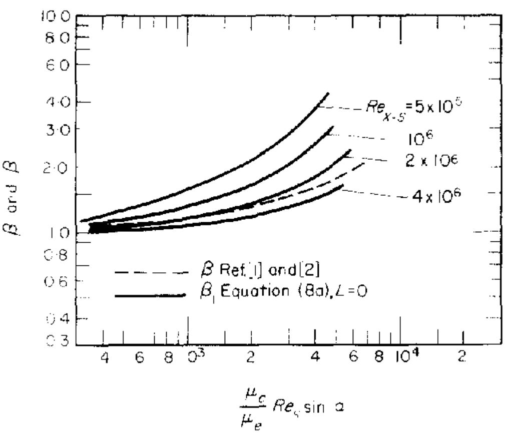

FIG. 1. Comparison of β [1,2] with β_1 (equation (8)).

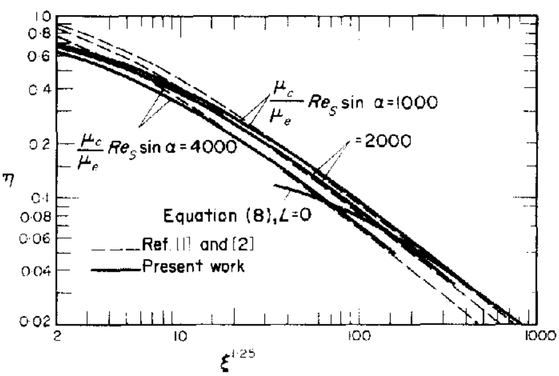

FIG. 2. Comparison of film cooling effectiveness of [1, 2] with the present study; $Re_{x_1} = 3\times10^6$.

DISCUSSION

It may be seen from Fig. 1 that the equations for β and β_1 yield nearly the same value when Re_{x-s} is in the neighborhood of 3×10^6. Since β is independent of Re_{x_1}, it would not take the value of unity as x approaches infinity. As can be seen in Fig. 1, this would give a discrepancy with β_1 of approximately 10 per cent for $Re_{x_1} = 4\times10^6$, and the difference would increase as Re_{x_1} increases. A compromise for a solution at the vicinity of the slot may be obtained by letting $Re_{x_1} = 3 \times 10^6$ and replacing the first term in the denominator of equation (8) by $1{\cdot}9\,Pr^{2/3}$. This replacement forces η to be unity at the edge of the slot, hence satisfying the initial temperature requirement. The adjusted solution is shown in Fig. 2 as a comparison with the results of [1]. The value of $\sin \alpha$ is taken to be unity, corresponding to normal injection. The agreement is quite satisfactory. The only discrepancy is at small values of ξ. Also, Re_{x_1} has been selected as a fixed quantity because the velocity profile in the boundary layer would approach a power-law profile depending on the value of Re_{x_1} rather than ξ. The value of n at $Re_{x_1} = 3 \times 10^6$ is approximately 5·5 [3] in the absence of secondary flow.

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SOLUTION OF THE NATURAL CONVECTION PROBLEM BY PARAMETER DIFFERENTIATION

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INTRODUCTION

THE PROBLEM of free convection over a semi-infinite isothermal flat plate has been analysed in the literature [1]. Such analysis involved a numerical solution of a boundary value problem which required iteration in order to satisfy the conditions stated at the boundaries for every value of Prandtl number, Pr. In the present note we present a non-iterative method, known as the method of parameter differentiation, to solve the same problem for various values of Pr. The method requires no iteration once a solution or the initial conditions of such a solution are known for one value of Pr. In the event a starting solution or conditions are not available, iteration is then required only for one value of Pr. Results for other values of Pr are then obtained by integrating the rate of change of the solution with respect to the parameter Pr. Each step in the calculation involves only a small perturbation in the parameter. By this approach, the equations solved are linear differential equations which can be solved noniteratively. Even though this method has been applied to the solution of simpler equations [2, 3], its application to the simultaneous nonlinear ordinary differential equations for the purpose of eliminating iteration is not evident in the literature. We like to note that Narayana and Ramamoorthy [4], in their analysis of the compressible boundary layer equations, attempted to eliminate iteration using the method of parameter differentiation. However, their attempt was not successful. This is due to the fact that they chose a two-parameter two-term superposition technique for their solution instead of a two-parameter three-term superposition like the one given by equations (7) and (8) in the present note. The choice of the number of terms in the solution, as it will become evident later on in the note, depends upon the number of the missing initial conditions in the solution.

ANALYSIS

The nonlinear ordinary differential equations governing the natural convection boundary layer flow over a semi-infinite isothermal flat plate in terms of similarity variables can be written in the usual notations as

$$F_{\eta\eta\eta} + 3FF_{\eta\eta} - 2F_\eta^2 + \theta = 0 \tag{1}$$

$$\theta_{\eta\eta} + 3PrF\theta_\eta = 0 \tag{2}$$

subject to the boundary conditions

$$\eta = 0: F(0) = F_\eta(0) = 0, \quad \theta(0) = 1$$
$$\eta = \infty: F_\eta(\infty) = 0, \quad \theta(\infty) = 0.$$

The subscript η implies differentiation with respect to η. In this note solutions are sought for different values of the parameter Pr. Differentiating equations (1) and (2) with respect to Pr we get

$$g_{\eta\eta\eta} + 3F_{\eta\eta}g + 3Fg_{\eta\eta} - 4F_\eta g_\eta + T = 0 \tag{3}$$

$$T_{\eta\eta} + 3F\theta_\eta + 3Prg\theta_\eta + 3PrFT_\eta = 0 \tag{4}$$

where

$$g = \frac{\partial F}{\partial Pr} \quad \text{and} \quad T = \frac{\partial \theta}{\partial Pr} \tag{5}$$

with the boundary conditions

$$\begin{aligned} \eta = 0&: g(0) = g_\eta(0) = 0, \quad T(0) = 0 \\ \eta = \infty&: g_\eta(\infty) = 0, \quad T(\infty) = 0. \end{aligned} \tag{6}$$

Equations (3) and (4) are now linear, and their solutions can be obtained by separating the dependent variables as

$$g = g_1 + \lambda g_2 + \mu g_3 \tag{7}$$

$$T = T_1 + \lambda T_2 + \mu T_3. \tag{8}$$